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The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

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Contents

Chapter 26

Quadratic Functions and Graphs - Grade 11

26.1 Introduction

In Grade 10, you studied graphs of many different forms. In this chapter, you will learn a little more about the graphs of functions.

26.2 Functions of the Form $y = a(x+p)^2 + q$

This form of the quadratic function is slightly more complex than the form studied in Grade 10, $y = ax^2 + q$. The general shape and position of the graph of the function of the form $f(x) = a(x+p)^2 + q$ is shown in Figure 26.1.

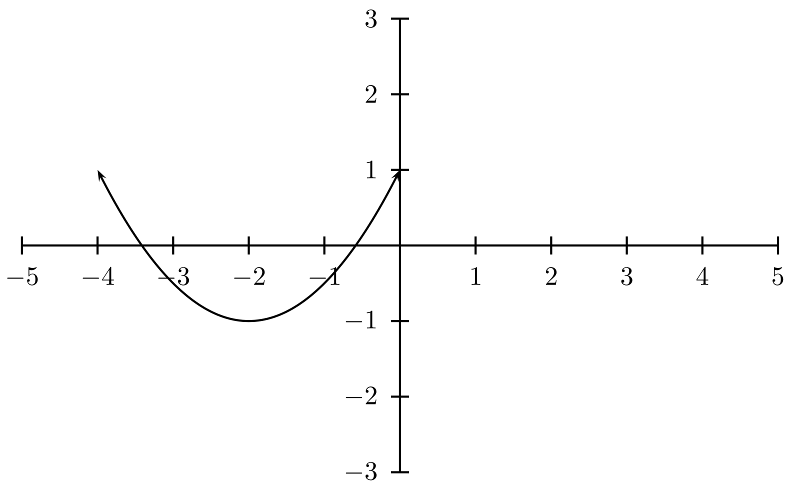


Figure 26.1: Graph of $f(x) = \frac{1}{2}(x+2)^2 - 1$

Activity :: Investigation : Functions of the Form $y = a(x+p)^2 + q$

1. On the same set of axes, plot the following graphs:
   A $a(x) = (x-2)^2$
   B $b(x) = (x-1)^2$
   C $c(x) = x^2$
   D $d(x) = (x+1)^2$

E $e(x) = (x + 2)^2$

Use your results to deduce the effect of p.

2. On the same set of axes, plot the following graphs:

   A $f(x) = (x - 2)^2 + 1$
   B $g(x) = (x - 1)^2 + 1$
   C $h(x) = x^2 + 1$
   D $j(x) = (x + 1)^2 + 1$
   E $k(x) = (x + 2)^2 + 1$

   Use your results to deduce the effect of q.

3. Following the general method of the above activities, choose your own values of p and q to plot 5 different graphs (on the same set of axes) of $y = a(x+p)^2+q$ to deduce the effect of a.

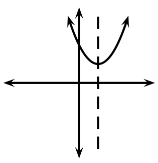
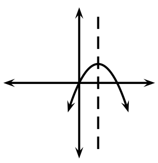
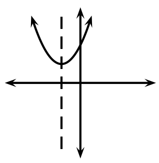
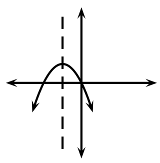
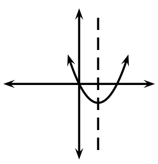
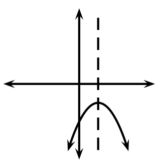
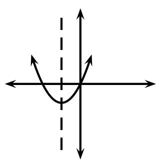
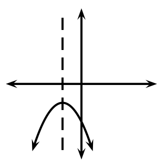
From your graphs, you should have found that a affects whether the graph makes a smile or a frown. If $a < 0$, the graph makes a frown and if $a > 0$ then the graph makes a smile. This is shown in Figure 10.9.

You should have also found that the value of p affects whether the turning point of the graph is above the x-axis ($p < 0$) or below the x-axis ($p > 0$).

You should have also found that the value of q affects whether the turning point is to the left of the y-axis ($q > 0$) or to the right of the y-axis ($q < 0$).

These different properties are summarised in Table 26.1. The axes of symmetry for each graph is shown as a dashed line.

Table 26.1: Table summarising general shapes and positions of functions of the form $y = a(x + p)^2 + q$. The axes of symmetry are shown as dashed lines.

	$p < 0$		$p > 0$	
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q \geq 0$				
$q \leq 0$				

26.2.1 Domain and Range

For $f(x) = a(x + p)^2 + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = a(x + p)^2 + q$ depends on whether the value for a is positive or negative. We will consider these two cases separately.

If $a > 0$ then we have:

$$\begin{aligned} (x+p)^2 &\geq 0 \quad \text{(The square of an expression is always positive)} \\ a(x+p)^2 &\geq 0 \quad \text{(Multiplication by a positive number maintains the nature of the inequality)} \\ a(x+p)^2+q &\geq q \\ f(x) &\geq q \end{aligned}$$

This tells us that for all values of x, $f(x)$ is always greater than q. Therefore if $a > 0$, the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$.

Similarly, it can be shown that if $a < 0$ that the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$. This is left as an exercise.

For example, the domain of $g(x) = (x-1)^2 + 2$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ can be calculated as follows:

$$\begin{aligned} (x-p)^2 &\geq 0 \\ (x+p)^2 + 2 &\geq 2 \\ g(x) &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

?

Exercise: Domain and Range

1. Given the function $f(x) = (x-4)^2 - 1$. Give the range of $f(x)$.
2. What is the domain the equation $y = 2x^2 - 5x - 18$?

26.2.2 Intercepts

For functions of the form, $y = a(x+p)^2 + q$, the details of calculating the intercepts with the x and y axis is given.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= a(x+p)^2 + q &(26.1) \\ y_{int} &= a(0+p)^2 + q &(26.2) \\ &= ap^2 + q &(26.3) \end{aligned}$$

If $p = 0$, then $y_{int} = q$.

For example, the y-intercept of $g(x) = (x-1)^2 + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} g(x) &= (x-1)^2 + 2 \\ y_{int} &= (0-1)^2 + 2 \\ &= (-1)^2 + 2 \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

The x-intercepts are calculated as follows:

$$\begin{aligned} y &= a(x+p)^2 + q &(26.4) \\ 0 &= a(x_{int}+p)^2 + q &(26.5) \\ a(x_{int}+p)^2 &= -q &(26.6) \\ x_{int} + p &= \sqrt{-\frac{q}{a}} &(26.7) \\ x_{int} &= \pm\sqrt{-\frac{q}{a}} - p &(26.8) \end{aligned}$$

However, (26.8) is only valid if $-\frac{q}{a} > 0$ which means that either $q < 0$ or $a < 0$. This is consistent with what we expect, since if $q > 0$ and $a > 0$ then $-\frac{q}{a}$ is negative and in this case the graph lies above the x-axis and therefore does not intersect the x-axis. If however, $q > 0$ and $a < 0$, then $-\frac{q}{a}$ is positive and the graph is hat shaped and should have two x-intercepts. Similarly, if $q < 0$ and $a > 0$ then $-\frac{q}{a}$ is also positive, and the graph should intersect with the x-axis.

For example, the x-intercepts of $g(x) = (x-1)^2 + 2$ is given by setting $y = 0$ to get:

$$\begin{aligned} g(x) &= (x-1)^2 + 2 \\ 0 &= (x_{int} - 1)^2 + 2 \\ -2 &= (x_{int} - 1)^2 \end{aligned}$$

which is not real. Therefore, the graph of $g(x) = (x-1)^2 + 2$ does not have any x-intercepts.

?

Exercise: Intercepts

1. Find the x- and y-intercepts of the function $f(x) = (x-4)^2 - 1$.
2. Find the intercepts with both axes of the graph of $f(x) = x^2 - 6x + 8$.
3. Given: $f(x) = -x^2 + 4x - 3$. Calculate the x- and y-intercepts of the graph of f.

26.2.3 Turning Points

The turning point of the function of the form $f(x) = a(x+p)^2 + q$ is given by examining the range of the function. We know that if $a > 0$ then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$ and if $a < 0$ then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.

So, if $a > 0$, then the lowest value that $f(x)$ can take on is q. Solving for the value of x at which $f(x) = q$ gives:

$$\begin{aligned} q &= a(x+p)^2 + q \\ 0 &= a(x+p)^2 \\ 0 &= (x+p)^2 \\ 0 &= x+p \\ x &= -p \end{aligned}$$

$\therefore x = -p$ at $f(x) = q$. The co-ordinates of the (minimal) turning point is therefore $(-p,q)$.

Similarly, if $a < 0$, then the highest value that $f(x)$ can take on is q and the co-ordinates of the (maximal) turning point is $(-p,q)$.

?

Exercise: Turning Points

1. Determine the turning point of the graph of $f(x) = x^2 - 6x + 8$.
2. Given: $f(x) = -x^2 + 4x - 3$. Calculate the co-ordinates of the turning point of f.
3. Find the turning point of the following function by completing the square: $y = \frac{1}{2}(x+2)^2 - 1$.

26.2.4 Axes of Symmetry

There is one axis of symmetry for the function of the form $f(x) = a(x + p)^2 + q$ that passes through the turning point and is parallel to the y-axis. Since the x-coordinate of the turning point is $x = -p$, this is the axis of symmetry.

?

Exercise: Axes of Symmetry

1. Find the equation of the axis of symmetry of the graph $y = 2x^2 - 5x - 18$
2. Write down the equation of the axis of symmetry of the graph of $y = 3(x - 2)^2 + 1$
3. Write down an example of an equation of a parabola where the y-axis is the axis of symmetry.

26.2.5 Sketching Graphs of the Form $f(x) = a(x + p)^2 + q$

In order to sketch graphs of the form, $f(x) = a(x + p)^2 + q$, we need to calculate determine four characteristics:

1. sign of a
2. domain and range
3. turning point
4. y-intercept
5. x-intercept

For example, sketch the graph of $g(x) = -\frac{1}{2}(x + 1)^2 - 3$. Mark the intercepts, turning point and axis of symmetry.

Firstly, we determine that $a < 0$. This means that the graph will have a maximal turning point.

The domain of the graph is $\{x : x \in \mathbb{R}\}$ because $f(x)$ is defined for all $x \in \mathbb{R}$. The range of the graph is determined as follows:

$$\begin{aligned}
(x+1)^2 &\geq 0 \\
-\frac{1}{2}(x+1)^2 &\leq 0 \\
-\frac{1}{2}(x+1)^2 - 3 &\leq -3 \\
\therefore f(x) &\leq -3
\end{aligned}$$

Therefore the range of the graph is $\{f(x) : f(x) \in (-\infty, -3]\}$.

Using the fact that the maximum value that $f(x)$ achieves is -3, then the y-coordinate of the turning point is -3. The x-coordinate is determined as follows:

$$\begin{aligned}
-\frac{1}{2}(x+1)^2 - 3 &= -3 \\
-\frac{1}{2}(x+1)^2 - 3 + 3 &= 0 \\
-\frac{1}{2}(x+1)^2 &= 0 \\
\text{Divide both sides by } -\tfrac{1}{2}\text{:}\quad (x+1)^2 &= 0 \\
\text{Take square root of both sides:}\quad x + 1 &= 0 \\
\therefore \quad x &= -1
\end{aligned}$$

The coordinates of the turning point are: $(-1, -3)$.

The y-intercept is obtained by setting $x = 0$. This gives:

$$\begin{aligned} y_{int} &= -\frac{1}{2}(0+1)^2 - 3 \\ &= -\frac{1}{2}(1) - 3 \\ &= -\frac{1}{2} - 3 \\ &= -\frac{1}{2} - 3 \\ &= -\frac{7}{2} \end{aligned}$$

The x-intercept is obtained by setting $y = 0$. This gives:

$$\begin{aligned} 0 &= -\frac{1}{2}(x_{int}+1)^2 - 3 \\ 3 &= -\frac{1}{2}(x_{int}+1)^2 \\ -3\cdot 2 &= (x_{int}+1)^2 \\ -6 &= (x_{int}+1)^2 \end{aligned}$$

which is not real. Therefore, there are no x-intercepts.

We also know that the axis of symmetry is parallel to the y-axis and passes through the turning point.

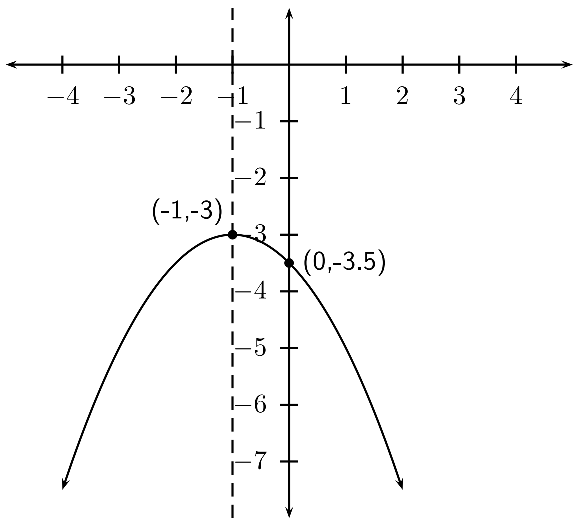


Figure 26.2: Graphs of the function $f(x) = -\frac{1}{2}(x_{int}+1)^2 - 3$

?

Exercise: Sketching the Parabola

1. Draw the graph of $y = 3(x-2)^2 + 1$ showing all the relative intercepts with the axes as well as the coordinates of the turning point.
2. Draw a neat sketch graph of the function defined by $y = ax^2 + bx + c$ if $a > 0$; $b < 0$; $b^2 = 4ac$.

26.2.6 Writing an equation of a shifted parabola

Given a parabola with equation $y = x^2 - 2x - 3$. The graph of the parabola is shifted one unit to the right. Or else the y-axis shifts one unit to the left. Therefore the new equation will become:

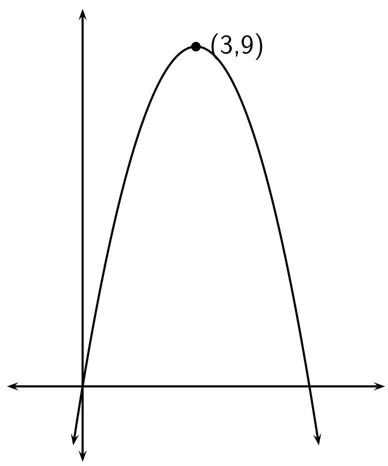
$$\begin{aligned} y &= (x-1)^2 - 2(x-1) - 3 \\ &= x^2 - 2x + 1 - 2x + 2 - 3 \\ &= x^2 - 4x \end{aligned}$$

If the given parabola is shifted 3 units down, the new equation will become:
(Notice the x-axis then moves 3 units upwards)

$$\begin{aligned} y + 3 &= x^2 - 2x - 3 \\ y &= x^2 - 2x - 6 \end{aligned}$$

26.3 End of Chapter Exercises

1. Show that if $a < 0$, then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty{,}q]\}$.
2. If (2;7) is the turning point of $f(x) = -2x^2 - 4ax + k$, find the values of the constants a and k.
3. The graph in the figure is represented by the equation $f(x) = ax^2 + bx$. The coordinates of the turning point are (3;9). Show that $a = -1$ and $b = 6$.



4. Given: $f : x = x^2 - 2x3$. Give the equation of the new graph originating if:

   A The graph of f is moved three units to the left.

   B The x - axis is moved down three.

5. A parabola with turning point (-1; -4) is shifted vertically by 4 units upwards. What are the coordinates of the turning point of the shifted parabola ?

Appendix A

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